Space-variant geometrical phases in focused cylindrical light beams

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Received May 11, 2007; revised August 4, 2007; accepted August 24, 2007;
posted September 6, 2007 (Doc. ID 82959); published October 10, 2007

We show that the depolarization caused when light is focused with a high-numerical-perture lens is accompanied by a space-variant geometrical phase. This phase results in the formation of modes with helicities and phase singularities that differ from those of the original beam. We show that this effect can be explained as a transverse shift of the rays, which is reminiscent of the recently discovered optical Hall–Magnus effect. Our results show that the asymmetric focal spot associated with the focus of linearly polarized light can be explained through geometrical effects. © 2007 Optical Society of America

OCIS codes: 260.5430, 110.2990, 350.1370.

It is well established that optical beams can carry angular momentum [1]. Recent studies have shown that in some instances the trajectory of a light beam can be affected by its angular momentum. This is because of a Berry phase that is added to the optical rays as they traverse different paths in momentum space [2,3] as the beam propagates through either an inhomogeneous medium or medium transitions. In some cases, this phase leads to the optical Hall effect (OHE) [2], in which circularly polarized beams undergo transverse shifts when refracted, and to weak anisotropy when a beam propagates in a smoothly inhomogeneous medium [3]. In this Letter we show how geometrical phases are manifested in tightly focused axisymmetric beams and how these phases can be interpreted through an effect that is analogous to the OHE.

For this analysis we consider a paraxial polarized beam propagating along the z axis with an electric field of the form

$$\mathbf{E}(\rho,\varphi,z) = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} E_0(\rho) e^{im\varphi} e^{-ikz},$$

where ρ is the radial coordinate around the optical axis, φ is the polar coordinate in the plane perpendicular to the beam axis, m is the topological charge, k is the wavenumber, and (α/β) is a complex unit vector representing the polarization of the beam.

Initially we assume that the beam is linearly polarized ($\alpha=1,\beta=0$) and that it is focused through a lens with a numerical aperture of $\sin\gamma$. The electric field in the image space of the lens can be found by using the Debye approximation [4,5], which with application of some trigonometry and identities relating to Bessel functions of the first kind takes on the form

$$\begin{bmatrix} e_x\\ e_y\\ e_z\end{bmatrix} = i^m \left\{\begin{matrix} -iA[I_m + 0.5(I_{m+2}e^{i2\varphi} + I_{m-2}e^{-i2\varphi})] \\ -A[0.5(I_{m+2}e^{i2\varphi} - I_{m-2}e^{-i2\varphi})] \\ -A[I_{m+1}e^{i\varphi} - I_{m-1}e^{-i\varphi}] \end{matrix}\right\} e^{im\varphi}. \tag{1}$$

In Eq. (1), A is a constant relating to the intensity of the beam, and

$$I_m(u,v) = \int_0^\gamma \cos^{1/2}\theta \sin\theta(1+\cos\theta) J_m\left(\frac{v\sin\theta}{\sin\gamma}\right) \times \exp(iu\cos\theta/\sin^2\gamma)\mathrm{d}\theta, \tag{2a}$$

$$I_{m\pm 2}(u,v) = \int_0^\gamma \cos^{1/2}\theta \sin\theta(1-\cos\theta) J_{m\pm 2}\left(\frac{v\sin\theta}{\sin\gamma}\right) \times \exp(iu\cos\theta/\sin^2\gamma)\mathrm{d}\theta, \tag{2b}$$

$$I_{m\pm 1}(u,v) = \int_0^\gamma \cos^{1/2}\theta \sin^2\theta J_{m\pm 1}\left(\frac{v\sin\theta}{\sin\gamma}\right) \times \exp(iu\cos\theta/\sin^2\gamma)\mathrm{d}\theta. \tag{2c}$$

Here u and v represent coordinates perpendicular and parallel to the optical axis in the image space, respectively, as defined in [5]. Note that the integration in Eqs. (2) is performed over the entire aperture of the lens and that θ is a function of ρ as defined in [5]. Analysis of Eqs. (1) and (2) shows that, in addition to a term with topological charge, m, the field in the image space also comprises terms with topological charges $m-2, m-1, m+1, m+2$. In the case of an incident plane wave ($m=0$), $I_{m+1}=I_1=-I_{-1}=-I_{m-1}$ and $I_{m+2}=I_2=I_{-2}=I_{m-2}$. Consequently the field in Eqs. (2) reduces to

$$\begin{bmatrix} e_x\\ e_y\\ e_z\end{bmatrix} = \left\{\begin{matrix} -iA[I_0 + I_2\cos 2\varphi] \\ -iAI_2 \sin 2\varphi \\ -2AI_1\cos\varphi \end{matrix}\right\},$$

which yields the well-known elongated focal spot associated with the focusing of linearly polarized beams [5]. When the incident beam carries a topological charge of $m=\pm 1,\pm 2$, the field in the image space comprises a component with topological charge $m=0$, which leads to anomalous bright centers in the focused beam as previously reported for Laguerre–Gauss beams [6].

To gain insight into the origin of the terms $I_{m\pm1,m\pm2}$ we consider the case of an incident beam with circular polarization. Using Eqs. (1) and (2) and following the logic presented in a previous publication on the tight-focusing of circularly polarized light [7] yields the electric field in the image space

$$\mathbf{E} = -i^{m+1}A\left(\begin{bmatrix}1\\ i\\ 0\end{bmatrix} I_m e^{im\varphi} + \begin{bmatrix}1\\ -i\\ 0\end{bmatrix} I_{m+2} e^{i(m+2)\varphi} - 2i\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} I_{m+1} e^{i(m+1)\varphi}\right). \quad (3)$$

The focused beam can be decomposed into three components, each with a uniform polarization: a component with the same topological charge and helicity as the original beam, a component with opposite helicity and a topological charge of $m+2$, and a component linearly polarized along the z axis with a topological charge of $m+1$. This decomposition is illustrated in Fig. 1, which shows the electric energy density in the focal plane for incident beams with different vorticities and ellipticities. We note that the decomposition shown in Eq. (3) yields three components with orthogonal polarizations, each with a different topological charge associated with their phase. However, $l+\sigma=m$ for each of these components. For any axisymmetric beam, $l+\sigma$ is equal to the angular momentum flux [8] through any plane perpendicular to the optical axis. This suggests that the spiral phases with topological charges $m+1$, $m+2$ shown in Eq. (3) are in some sense connected to the conservation of angular momentum flux in the system.

The observation that $l+\sigma=m$ for the three terms in Eq. (3) explains the form of the field when linearly polarized light is focused. Linearly polarized light is

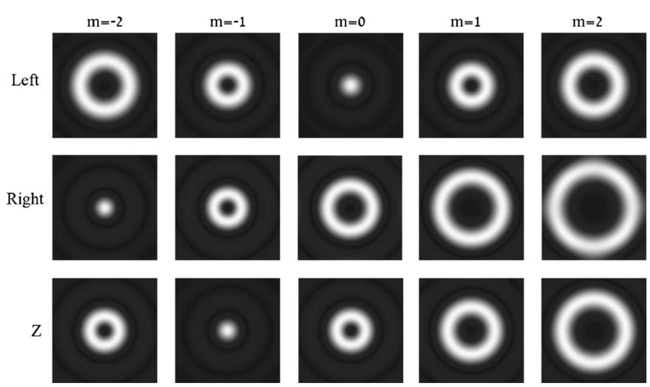

Fig. 1. Electric energy density associated with the left-hand polarized component (first row), the right-hand polarized component (second row), and the axial component (third row) within the focal spots of a 0.9 NA lens focusing left-hand polarized beams with different topological charges. Note that the left-hand polarized component typically contains about 100 times more energy than the right-hand polarized component and about 10 times more energy than the axial component.

the superposition of two components with opposite helicities. Focusing of the right-hand polarized component ($\sigma=1$) leads to a term with topological charge $m+2$ and $\sigma=-1$, and focusing of the component with left-hand circular polarization ($\sigma=-1$) leads to the term with topological charge $m-2$ and $\sigma=1$. Hence the asymmetry of the focal spot when linearly polarized light is focused can be predicted simply by enforcing a change in polarization and requiring that the total angular momentum flux of each component in the beam ($l+\sigma$) be conserved.

The accumulated phase associated with a polarized ray can arise from two sources: (a) the optical path that the ray traverses and (b) a geometrical contribution that arises from the path that the ray traverses in momentum space due to a change in the medium [3]. To establish that the spiral phases on the three components of Eq. (3) are of a geometrical origin, we calculate the phase between the field at two points in some plane perpendicular to the optical axis in the image space. For simplicity we choose two points located on the same circle around the optical axis (at points $\vec{r}_1=(r_0,0,z_0)$, $\vec{r}_2=(r_0,\varphi,z_0)$) and analyze the case of $m=0$. Since the fields at the two points of interest do not have the same polarization, we measure the phase by using Pancharatnam's definition of phase: $\phi_p=\arg\langle\mathbf{E}(\vec{r}_1),\mathbf{E}(\vec{r}_2)\rangle$ [9], where $\langle\rangle$ denotes an inner product and $\mathbf{E}(\vec{r}_1)$ and $\mathbf{E}(\vec{r}_2)$ are the electric fields at two different points on the wavefront. The calculation yields

$$\phi_p = \arg\langle\mathbf{E}(r,0),\mathbf{E}(r,\phi)\rangle = \varphi - \tan^{-1}\left[\frac{(I_0^2-I_2^2)\sin\varphi}{(I_0^2+I_2^2)\cos\varphi + 2I_1^2}\right].$$

Hence there is a nonzero phase between the fields at these two points. However, symmetry dictates that the optical paths traversed by two rays passing through these points are equal. Furthermore, a short calculation reveals that the phase between two points on either side of the exit pupil along a single ray is equal to the optical path traversed by the ray. Therefore the phase between the fields at two different points in the same plane after the exit pupil must arise from a geometrical effect. In particular, it must arise from the different geometrical transformations (rotations) that the two rays undergo when they are rotated at the lens interface and from the non-Abelian nature of the rotation group. This suggests that the formation of the modes with phase singularities $m\pm2$, $m\pm1$ is the result of a geometrical effect, which is associated with the bending of the rays. We note that space-variant geometrical phases have been discussed previously in the context of space-variant polarization state manipulations [9], but not in the context of tight focusing.

An alternative approach to defining the phase of a vector field is to use the rectifying phase. According to basic theory [10], the polarization of a complex three-dimensional field can be written as $\mathbf{E}(\mathbf{r}) = \exp(i\chi(\mathbf{r}))[\mathbf{A}(\mathbf{r})+i\mathbf{B}(\mathbf{r})]$, where $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are real vectors that are orthogonal to each other, and

$\chi(\mathbf{r})=1/2\arg(\mathbf{E}\cdot\mathbf{E})$ is the rectifying phase. Calculating this phase for the field shown in Eq. (3) (incident circular polarization) yields $\chi(\mathbf{r})=(m+1)\varphi=(l+\sigma)\varphi$. Thus, the vorticity of the rectifying phase of the focused beam is equal to the sum of vorticity and helicity of the incident beam. Figure 2 shows color maps of $2\chi(r)=\arg(\mathbf{E}\cdot\mathbf{E})$ at the focal plane for focused beams with different incident vorticities and different helicities. It is interesting to note the singularities in the rectifying phase that appear when the incident beam is not circularly polarized.

As a final point we show how the beam shape in the image space can be interpreted by using a description that is reminiscent of the recently discovered OHE [2,3]. The OHE predicts that when a circularly polarized ray (photon) alters its direction of propagation, it must undergo a transverse shift to conserve angular momentum. At the lens, the circularly polarized ray (photon) is refracted, and its spin-vector becomes a superposition of two orthogonally polarized states with opposite spin, and a degeneracy associated with polarization is removed. Consequently a single ray splits into two rays with opposite helicities. Conservation of angular momentum flux occurs if the total angular momentum associated with each of these rays is conserved. Hence the ray with flipped helicity undergoes a transverse shift of ΔL, which is manifested as an aberration $\Delta\varphi$ in the focal plane as shown in Fig. 3. The change in angular momentum associated with this transverse shift is the vector product of the shift vector and the wave vector, $\vec{k}$, which by conservation of angular momentum must equal 2 (because the spin of the ray has flipped from 1 to −1). Hence $\Delta\vec{L}\times\vec{k}=\Delta L\cdot 2\pi/\lambda=2$, which yields $\Delta L=\lambda/\pi$. Tracing all of the shifted rays reveals that they intersect the focal plane on a circle with a radius of λ/π around the focal spot. This yields an aberration, $\Delta\varphi=2\pi\Delta L/\lambda=2$. Integrating this aberration along a path connecting two points in the focal plane reveals that it is equivalent to a spiral phase of 2φ, which is consistent with the form of the beam depicted in Eq. (3). Thus the formation of components with helicities $m\pm 2$ can be described through a transverse shift in ray position that is reminiscent of the OHE.

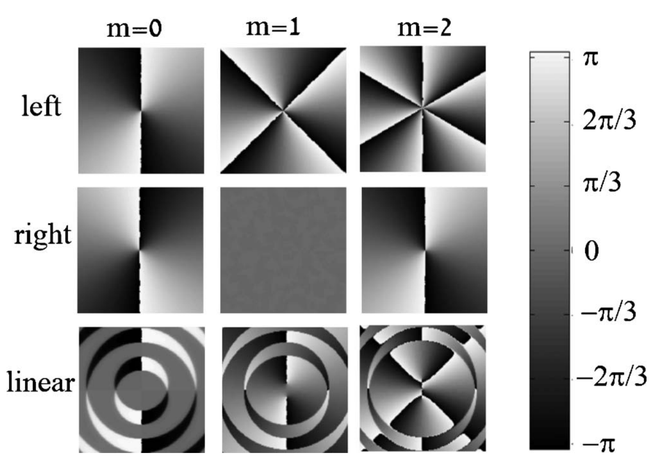

Fig. 2. Color maps showing $2\chi(r)=\arg(\mathbf{E}\cdot\mathbf{E})$ within the focal spots of a 0.9 NA lens focusing beams with left-hand circular, right-hand circular, and linear polarizations (rows) and vorticities $m=0,1,2$ (columns).

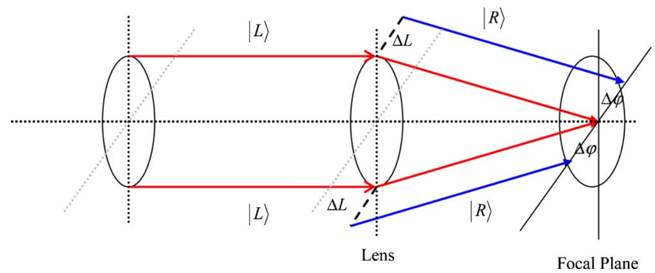

Fig. 3. (Color online) Illustration showing how the transverse shift of optical rays leads to the formation of modes with different topological charges. A left-hand polarized beam (inner, red rays) is incident on the lens. The rays are bent by the lens, removing a degeneracy associated with polarization, and two sets of rays with orthogonal helicity emerge. The rays with right-hand polarization (outer, blue) undergo a transverse shift (ΔL). The latter rays do not converge at the focal spot, and the shift is manifested as an aberration ($\Delta\varphi=2\varphi$), which leads to the formation of a mode with topological charge $m+2$.

To conclude, this Letter extends previous work on the tight focusing of beams carrying angular momentum [7,11] and provides a geometrical mechanism that can explain the structure of the beams in the image space. This Letter presents a link between previous experimental observations on the anomalous behavior of focused light and the OHE. In particular it demonstrates that the classical result regarding the elongated focal spot when linearly polarized plane is focused [5] can be explained through an OHE-like phenomenon.

This research was supported by a Marie Curie International Fellowship within the Sixth European Community Framework Program.

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